Chapter 1:
Systems of Linear Equations

Sec. 1.1-1.3:
Solving Systems of Linear Equations Using Elementary Row Operations

Solving Systems of Linear Equations

Recall…

Goal: Find all solutions to any system of linear equations (we will learn a procedure for this!)

Def: Two systems of equations are equivalent if they have exactly the same solutions set.

Idea: Keep replacing a system of linear equations with an equivalent one until eventually we end up with a system that is easy to solve.

Note: To save on the writing, we will use augmented matrices instead of writing out the entire system of equations each time (we don't have to keep writing the variables!)

Solving Systems of Linear Equations (Big Picture)

Solve…

$$\begin{aligned} 2x_1 + x_2 - 2x_3 &= 6 \\ x_1 - 2x_2 + 5x_3 &= 4 \\ 3x_1 - 5x_2 + 3x_3 &= 2 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= 4 \\ 2x_1 + x_2 - 2x_3 &= 6 \\ 3x_1 - 5x_2 + 3x_3 &= 2 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= 4 \\ 5x_2 - 12x_3 &= -2 \\ 3x_1 - 5x_2 + 3x_3 &= 2 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= 4 \\ 5x_2 - 12x_3 &= -2 \\ x_2 - 12x_3 &= -10 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= 4 \\ x_2 - 12x_3 &= -10 \\ 5x_2 - 12x_3 &= -2 \end{aligned}$$

$$\begin{aligned} x_1 - 19x_3 &= -16 \\ x_2 - 12x_3 &= -10 \\ 5x_2 - 12x_3 &= -2 \end{aligned}$$

$$\begin{aligned} x_1 - 19x_3 &= -16 \\ x_2 - 12x_3 &= -10 \\ 48x_3 &= 48 \end{aligned}$$

$$\begin{aligned} x_1 - 19x_3 &= -16 \\ x_2 - 12x_3 &= -10 \\ x_3 &= 1 \end{aligned}$$

$$\begin{aligned} x_1 &= 3 \\ x_2 - 12x_3 &= -10 \\ x_3 &= 1 \end{aligned}$$

$$\begin{aligned} x_1 &= 3 \\ x_2 &= 2 \\ x_3 &= 1 \end{aligned}$$

Solution Set = { (3,2,1) }

Solving Systems of Linear Equations (Big Picture)

<u>Solve…</u>

$$\begin{aligned} 2x_1 + \quad x_2 - 2x_3 &= \quad 6 \\ x_1 - 2x_2 + 5x_3 &= \quad 4 \\ 3x_1 - 5x_2 + 3x_3 &= \quad 2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & -2 & 6 \\ 1 & -2 & 5 & 4 \\ 3 & -5 & 3 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & 4 \\ 2 & 1 & -2 & 6 \\ 3 & -5 & 3 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & 4 \\ 0 & 5 & -12 & -2 \\ 3 & -5 & 3 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & 4 \\ 0 & 5 & -12 & -2 \\ 0 & 1 & -12 & -10 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & 4 \\ 0 & 1 & -12 & -10 \\ 0 & 5 & -12 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -19 & -16 \\ 0 & 1 & -12 & -10 \\ 0 & 5 & -12 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -19 & -16 \\ 0 & 1 & -12 & -10 \\ 0 & 0 & 48 & 48 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -19 & -16 \\ 0 & 1 & -12 & -10 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & -12 & -10 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{aligned} x_1 \qquad\qquad &= 3 \\ x_2 \qquad &= 2 \\ x_3 &= 1 \end{aligned}$$

Solution Set = { (3,2,1) }

Solving Systems of Linear Equations (Big Picture)

So we have to answer 2 questions…

1) What are we allowed to do to a system of linear equations to get an equivalent system of linear equations?
   (What's the augmented matrix version of this?)

2) What kind of "easy" system of linear equations are we trying to get to?
   (What's the augmented matrix version of this?)

We'll answer the 2nd question first

Solving "Easy" Systems of Linear Equations

Ex 3: Solve

$$\begin{aligned} x_1 &= 5 \\ x_2 &= -3 \\ x_3 &= 2 \end{aligned}$$

Ex 4: Solve (back substitution)

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 6 \\ x_2 - 4x_3 &= -6 \\ x_3 &= 1 \end{aligned}$$

Solving "Easy" Systems of Linear Equations

Ex 5: Solve

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + x_4 &= 9 \\ x_2 \qquad\qquad - 4x_4 &= 1 \\ 0 &= 5 \end{aligned}$$

(inconsistent)

Solving "Easy" Systems of Linear Equations

Ex 6: Solve

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + x_4 \qquad\qquad &= 6 \\ x_3 - 4x_4 + 2x_5 &= 3 \\ x_4 - 3x_5 &= 1 \\ 0 &= 0 \end{aligned}$$

(free variables, REF)

Solving "Easy" Systems of Linear Equations

Ex 7: Solve (free variables, RREF)

$$\begin{aligned} x_1 + 3x_2 \qquad\qquad\qquad\qquad &= 6 \\ x_3 \qquad + 2x_5 &= 3 \\ x_4 - 3x_5 &= 1 \\ 0 &= 0 \end{aligned}$$

What do we ultimately want in our final system of equations?

What do we want our final augmented matrix to look like?
ANS: We want the final augmented matrix to be in one of the following 2 forms…

REF – Row-Echelon Form
RREF – Reduced Row-Echelon Form

REF and RREF Matrices

Def: A matrix R is in row-echelon form (REF) if…

1) All rows of zeros are below all nonzero rows
2) The first number in every nonzero row is a 1 (called leading 1's)
3) Each leading 1 is to the right of the leading 1 in the row above it (leading 1's move down and to the right)
4) Each number below a leading 1 is 0

Ex's (REF):

REF and RREF Matrices

Def: A matrix R is in reduced row-echelon form (RREF) if…

1) All rows of zeros are below all nonzero rows
2) The first number in every nonzero row is a 1 (called leading 1's)
3) Each leading 1 is to the right of the leading 1 in the row above it (leading 1's move down and to the right)
4) Each number above and below a leading 1 is 0

Ex's (RREF):

REF and RREF Matrices

<u>Def</u>: A matrix *R* is in reduced row-echelon form (RREF) if…

1) All rows of zeros are below all nonzero rows
2) The first number in every nonzero row is a 1 (called leading 1's)
3) Each leading 1 is to the right of the leading 1 in the row above it (leading 1's move down and to the right)
4) Each number above and below a leading 1 is 0

$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving Systems of Linear Equations (Big Picture)

So we have to answer 2 questions…

1) What are we allowed to do to a system of linear equations to get an equivalent system of linear equations?
   (What's the augmented matrix version of this?)

2) What kind of "easy" system of linear equations are we trying to get to?
   (What's the augmented matrix version of this?)

Now we'll work on the 1st question

Elementary Operations (System of Equations Version)

Definition 1.1 Elementary Operations

The following operations, called ***elementary operations****, can routinely be performed on systems of linear equations to produce equivalent systems.*

I. Interchange two equations.

II. Multiply one equation by a nonzero *number.*

III. Add a multiple of one equation to a different *equation.*

Theorem 1.1.1

Suppose that a sequence of elementary operations is performed on a system of linear equations. Then the resulting system has the same set of solutions as the original, so the two systems are equivalent.

Elementary Row Operations (Augmented Matrix Version)

Definition 1.2 Elementary Row Operations

The following are called ***elementary row operations*** *on a matrix.*

I. Interchange two rows.

II. Multiply one row by a nonzero *number.*

III. Add a multiple of one row to a different *row.*

Note: Each elementary row operation is invertible and its inverse is also an elementary row operation. So, elementary row operations produce equivalent systems of equations.

Aside: Sets/When Are 2 Sets Equal?

Def: If A and B are 2 sets, A is a subset of B (denoted $A \subseteq B$) if every element of A is also an element of B.

I.e. $\forall\, x \in A\,,\, x \in B.$

Ex 8:

Let A be the solution set of $x^2 - 1 = 0$

Let B be the solution set of $x^3 - x = 0$

Find A and B. Is $A \subseteq B$?

Equal / incomparable

Aside: Sets/When Are 2 Sets Equal?

Def: If A and B are 2 sets, A is a subset of B (denoted $A \subseteq B$) if every element of A is also an element of B.

I.e. $\forall x \in A, x \in B$.

The most common way to show $A = B$ is to show…

1) $A \subseteq B$ and

2) $B \subseteq A$

Why not just find the sets?

Elementary Row Operations

Definition 1.2 Elementary Row Operations

The following are called ***elementary row operations*** *on a matrix.*

I. *Interchange two rows.*

II. *Multiply one row by a* nonzero *number.*

III. *Add a multiple of one row to a* different *row.*

Note: Each elementary row operation is invertible and its inverse is also an elementary row operation. So, elementary row operations produce equivalent systems of equations.

Elementary Row Operations

Note: Each elementary row operation is invertible and its inverse is also an elementary row operation. So, elementary row operations produce equivalent systems of equations.

REF and RREF

<u>RESULT</u>: Every matrix can be carried to a REF or RREF matrix by a sequence of elementary row operations.

Gaussian[3] Algorithm[4]

Step 1. If the matrix consists entirely of zeros, stop—it is already in row-echelon form.

Step 2. Otherwise, find the first column from the left containing a nonzero entry (call it a), and move the row containing that entry to the top position.

Step 3. Now multiply the new top row by $1/a$ to create a leading 1.

Step 4. By subtracting multiples of that row from rows below it, make each entry below the leading 1 zero.

This completes the first row, and all further row operations are carried out on the remaining rows.

Step 5. Repeat steps 1–4 on the matrix consisting of the remaining rows.

The process stops when either no rows remain at step 5 or the remaining rows consist entirely of zeros.

REF and RREF

<u>Ex 9</u>: Row reduce the following matrix to row-echelon form and reduced row-echelon form:

$$\begin{bmatrix} 0 & 0 & 2 & 1 & 5 \\ 0 & 0 & 5 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & 11 \end{bmatrix}$$

REF and RREF

Ex 10: Solve the following system of equations by row reducing its augmented matrix to row-echelon form.

$$\begin{aligned} 2x_1 + 6x_2 - 4x_3 &= 10 \\ 5x_1 - 4x_2 + \quad x_3 &= 6 \\ -2x_2 + 2x_3 &= -2 \end{aligned}$$

REF and RREF

Ex 11: Solve the following system of equations by row reducing its augmented matrix to reduced row-echelon form.

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 1 \\ 2x_1 - 4x_2 + x_3 \phantom{{}+ 3x_4} &= 5 \\ x_1 - 2x_2 + 2x_3 - 3x_4 &= 4 \end{aligned}$$

REF and RREF

Ex 12: Using a graphing calculator, solve the following system of equations by row reducing its augmented matrix to reduced row-echelon form.

$$\begin{aligned} 3x + y - 14z &= -1 \\ x \qquad + 10z &= -5 \\ 4x + y + 16z &= -1 \end{aligned}$$

REF and RREF

Gaussian Elimination

To solve a system of linear equations proceed as follows:

1. *Carry the augmented matrix to a reduced row-echelon matrix using elementary row operations.*
2. *If a row* $\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$ *occurs, the system is inconsistent.*
3. *Otherwise, assign the nonleading variables (if any) as parameters, and use the equations corresponding to the reduced row-echelon matrix to solve for the leading variables in terms of the parameters.*

Quick Definition: Rank of a Matrix

It turns out that given any matrix A, it's RREF is unique and its REF is not unique. But no matter if you reduce the matrix to REF or RREF, the number of leading 1's is always the same. This is called the rank of the matrix A.

Rank A = number of leading ones in any REF or RREF of matrix A

Definition 1.4 Rank of a Matrix

The ***rank*** *of matrix A is the number of leading 1s in any row-echelon matrix to which A can be carried by row operations.*

Rank of a Matrix

Ex 13: Compute the rank of $A = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & -5 & 8 \end{bmatrix}$

Rank of a Matrix

Theorem 1.2.2

Suppose a system of m equations in n variables is ***consistent****, and that the rank of the augmented matrix is r.*

1. *The set of solutions involves exactly $n-r$ parameters.*
2. *If $r < n$, the system has infinitely many solutions.*
3. *If $r = n$, the system has a unique solution.*

Quick Definition: Homogeneous Systems of Linear Equations

Def: A system of linear equations is homogeneous if the constant term in each equation is 0.

An example of a homogeneous system of linear equations…

$$\begin{aligned} 4x_1 + 2x_2 - 3x_3 + 6x_4 &= 0 \\ 3x_1 - 5x_2 + 2x_3 - 8x_4 &= 0 \\ 7x_1 + 3x_2 - 8x_3 + 5x_4 &= 0 \end{aligned}$$

Notes:

- A homogeneous system of linear equations always has at least 1 solution called the trivial solution. This is the one where all variables have value 0.
- If a homogeneous system of linear equations has a solution where some variable's value is not zero, we call this a nontrivial solution.

Homogeneous Systems of Linear Equations

Ex 14: Find a basic set of solutions and all solutions to the homogeneous system of linear equations with the following coefficient matrix…

$$A = \begin{bmatrix} 1 & -3 & 0 & 2 & 2 \\ -2 & 6 & 1 & 2 & -5 \\ 3 & -9 & -1 & 0 & 7 \\ -3 & 9 & 2 & 6 & -8 \end{bmatrix}$$

Homogeneous Systems of Linear Equations

The existence of a nontrivial solution in Example 1.3.1 is ensured by the presence of a parameter in the solution. This is due to the fact that there is a *nonleading* variable (x_3 in this case). But there *must* be a nonleading variable here because there are four variables and only three equations (and hence at *most* three leading variables). This discussion generalizes to a proof of the following fundamental theorem.

Theorem 1.3.1

If a homogeneous system of linear equations has more variables than equations, then it has a nontrivial solution (in fact, infinitely many).

Homogeneous Systems of Linear Equations

Theorem 1.3.2

Let A be an $m \times n$ matrix of rank r, and consider the homogeneous system in n variables with A as coefficient matrix. Then:

1. *The system has exactly $n - r$ basic solutions, one for each parameter.*
2. *Every solution is a linear combination of these basic solutions.*